Unit - I

Gaussian Wave Packet and it's Property for free Particle :-

// The wave Packet for which the Product of uncertainity in Position and momentum is minimum is called Gaussian Wave Packet.

i.e, for a Gaussian Wave Packet travelling along +ve x direcⁿ we will have, $\Delta x \, \Delta P_x = \frac{\hbar}{2}$ (min)

Let, ψ be the Wave funⁿ representing Gaussian wave Packet.

we get, two funⁿ f and ϕ obtain from when we applied momentum and Position operator. i.e,

$$f = \hat{P}\psi = -i\hbar \frac{\partial \psi}{\partial x} \longrightarrow (1)$$

and $$\phi = \hat{x}\psi \longrightarrow (11).$$

f and ϕ must be related to each other

$$\phi = \alpha f \longrightarrow (3) \qquad \text{where } \alpha = \text{const.}$$

Consider, $\int f^* \phi \, dx = \int i\hbar \frac{\partial \psi^*}{\partial x} x\psi \, dx.$

$\Rightarrow \int f^* \alpha f \, dx = \int i\hbar \frac{\partial \psi^*}{\partial x} x\psi \, dx.$

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If funⁿ 'f' is normalized then $\int f^* f \, dx = 1$

then above eq^n becomes.

$$\alpha = \int i\hbar \frac{\partial \psi^*}{\partial x} x \psi \, dx.$$

Thus, α is an imaginary const. so eq^n (3) can be rewritten as

$$\phi = \pm i\alpha f \longrightarrow (4).$$

using (2) and (1) in (4) we get,

$$x\psi = \pm i\alpha\left(-i\hbar \frac{\partial \psi}{\partial x}\right)$$

$$\Rightarrow x\psi = \pm \alpha\hbar \frac{\partial \psi}{\partial x}$$

$$\Rightarrow \int \frac{\partial \psi}{\psi} = \pm \frac{1}{\hbar\alpha}\int x \, dx$$

$$\Rightarrow \ln\psi = \pm \frac{1}{\hbar\alpha}\frac{x^2}{2} + \text{const.}$$

$$\Rightarrow \psi = e^{\pm \frac{1}{\hbar\alpha}\frac{x^2}{2} + c}$$

$$\therefore \psi = e^{c} e^{\pm \frac{1}{\hbar\alpha}\frac{x^2}{2}}$$

$$\therefore \psi = N e^{\pm \beta^2 x^2} \longrightarrow (5), \quad \beta^2 = \frac{1}{2\hbar\alpha} \longrightarrow (6).$$

$$e^{c} = N \text{ (say)}$$

if exponential is +ve

i.e. $\psi = N e^{+\beta^2 x^2}$ then for $x \to \pm\infty$

$\psi \to \infty$, which is not possible.

So, ~~for~~ $\psi = N e^{-\beta^2 x^2} \to (7)$.

~~Now~~ Now using normalization condition,

$$\int_{-\infty}^{\infty} \psi^* \psi \, dx = 1$$

$$\Rightarrow \int_{-\infty}^{\infty} N^2 e^{-2\beta^2 x^2} dx = 1 \longrightarrow (8)$$

Let $2\beta^2 x^2 = y^2 \longrightarrow$ (9) $\Rightarrow y = \sqrt{2}\beta x$

Diff^n. w.r.to x.

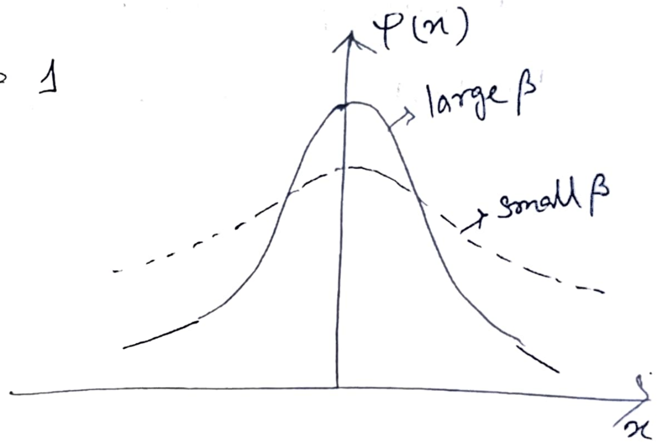
$\Rightarrow 2\cdot 2\beta^2 x\, dx = 2y\,dy$

$\Rightarrow 2\beta^2 x\, dx = y\,dy$

$$\Rightarrow dx = \frac{\sqrt{2}\beta x\, dy}{2\beta^2 x}$$

$$dx = \frac{dy}{\sqrt{2}\beta} \longrightarrow (10).$$

using (9) and (10) in eqn (8).



$$\Rightarrow \frac{N^2}{\sqrt{2}\beta}\int_{-\infty}^{\infty} e^{-y^2}\,dy = 1$$

$$\Rightarrow \frac{N^2}{\sqrt{2}\beta}\sqrt{\pi} = 1$$

$$\Rightarrow N^2 = \frac{\sqrt{2}\beta}{\sqrt{\pi}}$$

$$\Rightarrow N = \frac{2^{1/4}\beta^{1/2}}{\pi^{1/4}}$$

$$\text{So, } \psi = \left(\frac{2\beta^2}{\pi}\right)^{1/4} e^{-\beta^2 x^2}$$

this represents Gaussian wavefunction for Gaussian wave packet.

Property :-

(1) Any wave packet which is associated with a particle in motion for which $\Delta x \Delta p = \hbar/2$ (min) which is similar to Gaussian wave.

(2) $\psi(x)$ vs x graph is symetric about y-axis.

(3) the maximum value of $[f(x)]_{max} = \left(\frac{2\beta^2}{\pi}\right)^{1/4}$

(4) As β increases the peak of the wave become narrow and vice versa.

(5) at $x = \pm\frac{1}{\beta}$, the amplitude falls symmetrically to $\frac{1}{e}N = 37\%$ of N.

(6) For large β the fall of Amplitude is faster and vice versa.